

Factorization of quantum charge transport for noninteracting fermions

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We show that the statistics of the charge transfer of noninteracting fermions through a two-lead contact is generalized binomial at any temperature and for any form of the scattering matrix: an arbitrary charge-transfer process can be decomposed into independent single-particle events. This result generalizes previous studies of adiabatic pumping at zero temperature and of transport induced by bias voltage.

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I. INTRODUCTION

Noise is usually considered as an unwanted characteristic of electronic circuits. Sometimes, however, studying electronic noise gives important information about physical systems. In particular, noise in small junctions is affected by the fermionic statistics of electrons and is an essential part of quantum mesoscopic transport (see, e.g., Ref. 1). Even more information about the quantum behavior of charge carriers is encoded in the full probability distribution of the transferred charge: a concept known as full-counting statistics.²

The full-counting statistics of electrons is most studied for the simplest model of noninteracting fermions in various setups. In this model, the general expression for the probability distribution of the transferred charge is given by an elegant determinant formula of Levitov and Lesovik.²⁻⁵ This formula expresses the generating function for the probability distribution as a certain functional determinant involving the single-particle scattering matrix. An exact calculation of this functional determinant may be performed in several special setups of the charge transfer,²⁻⁷ and until recently most of the results on the full-counting statistics in this model concentrated on studying such exactly solvable cases.

However in the recent years a progress has been made in understanding general properties of charge transfer encoded in the determinant formula. Namely, it has been shown that, in the case of a contact with two external leads, the total electronic transfer is given by a superposition of uncorrelated elementary charge transfers of single electrons. In each of such single-electron events, one electron has a certain probability p to pass through the contact (thereby transferring one quantum of charge) and the probability $1-p$ to reflect (resulting in no charge transfer). This decomposition into elementary charge-transfer events (dubbed “generalized binomial” statistics⁸) has been shown under various assumptions: at zero temperature for a charge transfer driven by a time-dependent bias voltage,⁹ at zero temperature for the adiabatic-pumping problem,¹⁰ and in the wave-packet formalism (a finite number of wave packets with a momentum-dependent scattering).⁸

The goal of the present paper is to lift the assumptions made in those previous works and to show that the generalized binomial statistics is universally valid for any charge-transfer problem involving noninteracting fermions: at any temperature and for any time- and energy-dependent scat-

terer. A necessary condition remains that the initial state does not involve any entanglement between the leads or between different charge sectors in each lead.

It may be therefore practical to describe the statistics for noninteracting fermions by the distribution $\mu(p)$ of the effective transparencies of the elementary charge transfers instead of the probabilities of the total transfer. To illustrate this construction, we list several known examples where the distribution $\mu(p)$ is exactly known. Finally, to illustrate the importance of the noninteracting assumption, we also present several examples of *interacting* systems where the full-counting statistics is *not* generalized binomial. In those cases, we may interpret the effect of interaction as a shift of the transparencies p to the complex plane.

The paper is organized as follows. First, to fix the notation, we briefly review the derivation of the Levitov-Lesovik determinant formula in Sec. II. In Sec. III we obtain our main result: a decomposition of an arbitrary charge-transfer process into independent tunneling events. In Secs. IV and V, we illustrate our construction with several noninteracting and interacting examples, respectively. Finally, we conclude with Sec. VI, where we discuss possible generalizations and applications of our results. Some technical details are relegated to Appendixes A and C.

II. GENERATING FUNCTION AND DETERMINANT FORMULA

In this section we briefly review some known results from the theory of full-counting statistics for noninteracting fermions (mainly following the method in Ref. 11). We also introduce the notation and specify the assumptions used in the subsequent sections.

A. Generating function for full-counting statistics

A typical full-counting-statistics problem involves determining the probabilities of a given charge transfer through the junction. In this paper, we will consider the case of a junction with two leads, and therefore the definitions in this section will be written for a two-lead junction even though they admit a natural generalization to a many-lead situation.¹² In the case of two leads, the charge transfer is characterized by one number q : an integer number of electrons moved from the left to the right lead (or $-q$ electrons

from the right to the left lead). The ingredients of the problem are as follows: (i) the initial density matrix of the system $\hat{\rho}_0$ and (ii) the unitary evolution operator \hat{U} (over the time when the measurement is performed). To formally define the transferred charge q , we also need (iii) an operator \hat{Q} which counts the particles in one of the leads (say, the right lead).

Then one easily sees that the probabilities of a given charge transfer q can be found from the generating function

$$\chi(\lambda) = \text{Tr}(\hat{\rho}_0 \hat{U}^\dagger e^{i\lambda \hat{Q}} \hat{U} e^{-i\lambda \hat{Q}}) / \text{Tr} \hat{\rho}_0, \quad (1)$$

where λ is an auxiliary variable and the trace is taken in the many-body Fock space (for technical reasons, it is convenient to explicitly include the normalization of the density matrix). For a good definition of charge-transfer *probabilities*, one needs to further assume that different charge sectors are not entangled, i.e., that $\hat{\rho}_0$ commutes with \hat{Q} . Under this assumption one finds that thus defined $\chi(\lambda)$ may indeed be interpreted as a generating function,

$$\chi(\lambda) = \sum_{q=-\infty}^{\infty} P_q e^{i\lambda q}, \quad (2)$$

for some probabilities $0 \leq P_q \leq 1$. The periodicity of $\chi(\lambda)$ (or, equivalently, the quantization of q) follows from the fact that the spectrum of \hat{Q} is integer. It will therefore be convenient for our purposes to treat $\chi(\lambda)$ as a function of the complex variable

$$u = e^{i\lambda}. \quad (3)$$

Then Eq. (2) becomes the Laurent series of the complex function $\chi(u)$. The Fourier components of $\chi(\lambda)$ (the coefficients of the Laurent series) define the probabilities P_q , while the derivatives at $\lambda=0$ give the moments of the charge transfer $\langle q^n \rangle$.

In the case of a periodic pumping, it may be useful to define a “full-counting statistics per period”⁵ as the extensive part of $\chi(\lambda)$,

$$\chi_0(\lambda) = \exp \left[\lim_{N_p \rightarrow \infty} \frac{1}{N_p} \ln \chi(\lambda) \right], \quad (4)$$

where the full-counting statistics $\chi(\lambda)$ is collected over N_p periods. Defined in this way, the characteristic function $\chi_0(\lambda)$ may be used for calculating extensive quantities (such as cumulants of the transferred charge), but the effects associated with the beginning and ending of the measurement^{2,4} are neglected.

B. Determinant formula for noninteracting fermions

In the case of noninteracting fermions, general expression (1) may be simplified and rewritten in terms of *single-particle* operators: the so-called Levitov-Lesovik determinant formula.²⁻⁵ In this section we briefly present its derivation following the approach in Ref. 11.

For noninteracting fermions, the multiparticle operators (acting in the Fock space) are related to the corresponding single-particle operators (acting in the Hilbert space of one

particle) with the help of the second-quantized formalism. For notational convenience, we will use the “hatted” notation [as in Eq. (1)] for operators acting in the Fock space, while the “unhatted” letters will denote the corresponding single-particle operators. For quantum-mechanical observables (such as the charge operator), the correspondence is given by $\hat{Q} = \psi^\dagger Q \psi$ (where ψ^\dagger and ψ are the fermionic creation and annihilation operators, and the summation over single-particle states is assumed), while for their exponentials (such as the evolution operator or the density matrix) the correspondence is $\hat{U} = \exp[\psi^\dagger (\ln U) \psi]$ (see Appendix A for details).

Now we are ready to derive the Levitov-Lesovik determinant formula, starting from Eq. (1) and making the following assumptions:

(1) *The operator \hat{Q} is a fermionic bilinear.* This is obviously true for the operator of the charge in the right lead. However, more generally, one can use the same formalism for studying full-counting statistics for other physical quantities (e.g., spin).

(2) *The evolution operator \hat{U} is an exponential of a fermionic bilinear.* This is equivalent to requiring that the Hamiltonian is a fermionic bilinear (see Appendix A), i.e., that the fermions are noninteracting.

(3) *The density matrix $\hat{\rho}_0$ is an exponential of a fermionic bilinear.* $\hat{\rho}_0 = e^{-\hat{h}_0}$, where $\hat{h}_0 = \psi^\dagger h_0 \psi$ is a Hermitian fermionic bilinear [note that, if using Eq. (1), we do not need to normalize the density matrix]. Usually, one considers a thermal equilibrium with h_0 given by the physical Hamiltonian divided by the temperature. However, our assumption is much less restrictive. For example, one can take an initial state with different temperatures of the leads and/or of different channels or even prepare an equilibrium state using a different Hamiltonian.

(4) *The operators \hat{Q} and $\hat{\rho}_0$ commute.* This condition expresses the absence of charge entanglement in the initial state and is a necessary condition for interpreting $\chi(\lambda)$ as a generating function for charge-transfer probabilities.

The trace of the product of exponentials of fermionic bilinears in Eq. (1) can be rewritten as a determinant of an operator acting in the single-particle Hilbert space (see Ref. 11 and Appendix A for details),

$$\chi(\lambda) = \det[1 + n_F(U^\dagger e^{i\lambda Q} U e^{-i\lambda Q} - 1)]. \quad (5)$$

Here U is a single-particle evolution operator and

$$n_F = \frac{\rho_0}{\rho_0 + 1} = \frac{1}{1 + e^{h_0}} \quad (6)$$

is the occupation-number operator.

C. Regularization of the determinant

Depending on the dimensionality and on the asymptotic behavior of the operators involved in Eq. (5), the determinant may require a regularization. If the dimension of the single-particle Hilbert space is finite (this is the case in the wave-packet formalism in Ref. 8), all the matrices in Eq. (5) are

finite dimensional and no regularization is needed.

If the single-particle Hilbert space has an infinite dimension, then a regularization may be needed or not, depending on the properties of the Hamiltonians describing the evolution (in the operator \hat{U}) and the initial state (in the operator $\hat{\rho}$). Usually, one considers a certain dispersion of the particle in the leads (the same in the evolution and the initial-state Hamiltonians), with the fermions filling the states up to some Fermi energy. Determinant (5) is well defined if the matrix tends to unity sufficiently fast at both large positive and negative energies. At large positive energies, n_F tends to zero, and the matrix under determinant tends to unity. At large negative energies, the determinant is cut off either by the Fermi energy or by the energy dependence of U : at very low energy, the particle cannot penetrate through the barrier, then U commutes with Q , and the matrix again tends to unity. So determinant (5) is well defined if one uses exact dispersion relation and the energy dependence of the scattering matrix.

The only (but historically very common in literature) case when the determinant needs a regularization is the approximation where the spectrum is linearized (so that the Fermi energy does not serve as a regularizing parameter) and the scattering matrix is assumed to be energy independent (“instant-scattering” or “adiabatic pumping” limit).^{3,13,14} In that case, the matrix in Eq. (5) is infinite dimensional and tends to a nonunity matrix in the negative-energy asymptotics; therefore the determinant needs to be regularized. The regularization may be performed by admitting a weak energy dependence of U , so that the determinant converges, and then letting the corresponding energy scale tend to infinity. Technically, this procedure amounts to simply re-expressing the determinant in a manifestly convergent form where the weak energy dependence of U becomes inessential. This approach was used in Ref. 14 and then more generally (but also more formally) in Ref. 15. For completeness, we review the details of adiabatic approximation in Appendix B.

Therefore, we assume, without loss of generality, that an appropriate energy dependence is already included in the evolution operator U , and determinant (5) does not need any further regularization. Moreover, in the future discussion we will be interested in the zeros of the determinant, which are regularization independent.

III. DECOMPOSITION INTO SINGLE-PARTICLE EVENTS

A. Zeros of the generating function

To demonstrate that the full-counting statistics for noninteracting fermions can be decomposed into elementary single-particle transfer events, we first re-express determinant (5) in terms of the spectral properties of a single operator, whose eigenvalues give transmission probabilities of those elementary transfers.

This transformation proceeds as follows. Using the fact that the charge operator Q is a projector ($Q^2=Q$), we rewrite $e^{i\lambda Q}=1+(e^{i\lambda}-1)Q$ in Eq. (5) in order to obtain, after some simple algebra,

$$\chi(\lambda) = \det([1 + (e^{i\lambda} - 1)X]e^{-i\lambda Q}). \quad (7)$$

Here we have introduced a new operator,

$$X = (1 - n_F)Q + n_F U^\dagger Q U. \quad (8)$$

One can see that the spectrum of the operator X is

- (1) real and
- (2) confined to the interval $[0,1]$.

Both properties become obvious if one considers a Hermitian operator conjugate to X by a similarity transformation,

$$\tilde{X} = (n_F)^{-1/2} X (n_F)^{1/2} = (1 - n_F)Q + (n_F)^{1/2} U^\dagger Q U (n_F)^{1/2} \quad (9)$$

(we have used our assumption $[n_F, Q]=0$; n_F is Hermitian and positive definite so that $(n_F)^{\pm 1/2}$ is well defined¹⁶).

The reality of the spectrum follows from the hermiticity of \tilde{X} . The constraint on the eigenvalues follows from the observation that both \tilde{X} and $1-\tilde{X}$ are represented as sums of two positive operators [see Eq. (9) and the one obtained from it by replacing $Q \rightarrow 1-Q$].

The spectral properties of the operator X immediately imply a constraint on the positions of zeros of the generating function $\chi(\lambda)$ analytically continued to the complex plane of the variable $u=e^{i\lambda}$,

$$\chi(u) = \det([1 + (u - 1)X]u^{-Q}). \quad (10)$$

The zeros of $\chi(u)$ can be 0, ∞ (from the factor u^{-Q}), or coincide with $1-X_n^{-1}$, where X_n are the eigenvalues of the operator X . From the spectral properties of X , it then follows that zeros of $\chi(u)$ are confined to the negative real axis of the complex u plane. This conclusion has been previously reached in Ref. 10 under the assumptions of zero temperature and instant scattering. Generalizing it to arbitrary temperatures and to arbitrary evolution operators (under the assumptions formulated in Sec. II) constitutes the main result of our present paper.

B. Generalized binomial statistics and effective transparencies

Expression (10) may be further interpreted in terms of decomposing the charge transfer into a superposition of independent tunneling events, with the eigenvalues of X giving the probabilities of those tunneling processes. To make this interpretation more explicit, we get rid of the last factor $\exp[-i\lambda Q]$ in Eq. (10) by¹⁷

$$\begin{aligned} \chi(\lambda) &= \det([1 + (e^{-i\lambda} - 1)X]e^{-i\lambda X}e^{i\lambda X}e^{-i\lambda Q}) \\ &= e^{i\lambda \text{tr}(X-Q)} \det([1 + (e^{i\lambda} - 1)X]e^{-i\lambda X}) \\ &= e^{i\lambda \langle q \rangle} \prod_n e^{-i\lambda X_n} [1 + (e^{i\lambda} - 1)X_n], \end{aligned} \quad (11)$$

where in the last line the product is taken over the eigenvalues of the operator X , and

$$\langle q \rangle = -i \partial_\lambda \ln \chi(\lambda)|_{\lambda=0} = \text{tr}(X - Q) = \text{tr}[n_F(U^\dagger Q U - Q)] \quad (12)$$

is the average total transferred charge.¹⁵

Each factor $[1 + (e^{i\lambda} - 1)X_n]$ in Eq. (11) corresponds to an elementary single-electron event with the transmission probability X_n and reflection probability $1-X_n$. The other expo-

nential factors in Eq. (11) correspond to a deterministic “background” charge transfer, which does not produce any noise.

Such a statistics given by a superposition of individual transmission events with different transparencies was dubbed *generalized binomial statistics* in Ref. 8. Note that the effective transparencies are given by the spectrum of the *effective-transparency operator* X and depend in a complicated way on the time dynamics of the junction and on the initial thermodynamic state.¹⁸

C. Spectrum of the effective-transparency operator

Using the result derived above, we may now describe the full-counting statistics of noninteracting fermions by the distribution of effective transparencies (the density of eigenvalues of the operator X) instead of the probabilities P_q . Following Ref. 19, we define the effective-transparency density as

$$\mu(p) = \text{tr } \delta(p - X). \quad (13)$$

Since the spectrum of X defines the density of zeros of $\chi(u)$ on the negative real semiaxis of u plane, the density of states $\mu(p)$ may be related to the jump of the derivative of $\chi(\lambda)$ at the corresponding point. A simple calculation gives¹⁹

$$\begin{aligned} \mu(p) &= - \frac{1}{2\pi p(1-p)} \partial_\lambda \ln \chi(\lambda) \Big|_{u-i0}^{u+i0} \\ &= \frac{1}{\pi} \text{Im } \partial_p \ln \chi(p - i0), \end{aligned} \quad (14)$$

where

$$p = \frac{1}{1 - e^{i\lambda}} = \frac{1}{1 - u}. \quad (15)$$

Conversely, knowing the spectral density $\mu(p)$, one can reconstruct the cumulant generating function $\ln \chi(\lambda)$ (up to an overall deterministic charge transfer) by using Eq. (7),

$$\ln \chi(\lambda) = -i\lambda \text{Tr } Q + \int_0^1 dp \mu(p) \ln[1 + (e^{i\lambda} - 1)p]. \quad (16)$$

As for the structure of the spectrum of effective transparencies, there are two alternatives: the spectrum may be either discrete or continuous. An analysis of available examples gives the following possibilities:

(1) In a finite-dimensional problem (e.g., a finite number of incident wave packets⁸), the matrix is finite dimensional, and the spectrum of effective transparencies is obviously finite and discrete.

(2) At zero temperature, in the instant-scattering approximation, for a continuous and periodic time dependence of the scattering matrix,¹⁰ the spectrum is discrete, with possible accumulation points at $p=0$ and $p=1$.

(3) At zero temperature, in the instant-scattering approximation, for a discontinuous time dependence of the scattering matrix, the spectrum typically becomes continuous.¹⁹

(4) From the above two examples, one can deduce that at zero temperature, in the instant-scattering approximation, for

a continuous nonperiodic time dependence (a finite-time pumping pulse), the spectrum is discrete if the scattering matrix returns to its initial value and continuous otherwise. This can be shown by a linear fractional mapping of the time axis on a circle,⁵ thus reducing the nonperiodic problem to a periodic one, with either continuous or discontinuous time dependence of the scattering matrix.

(5) For finite temperatures or energy-dependent scattering, the spectrum of X is typically continuous. This can be seen in the examples presented in Sec. IV. The physical reason for a continuous spectrum is that particles incident at different energies have different effective transparencies, and thus integration over energies smears the spectrum.

If the spectrum $\mu(p)$ is continuous, the singularities form a branch cut of the function $\chi(u)$ along the negative real axis (or along a part of it). At this branch cut, the function $\chi(u)$ is discontinuous, with the jump obeying Eq. (14). A more detailed analysis of the spectrum of the effective transparencies will be presented in a subsequent publication.²⁰

Note that in the case of zero temperature and adiabatic pumping, the spectrum of effective transparencies may alternatively be found from a *different* operator labeled n_F^z in Ref. 10 and M in Ref. 19. It is worth emphasizing that although the two operators are different, at zero temperature and in the instant-scattering approximation their spectra coincide, as shown in Appendix C.

IV. EXAMPLES: NONINTERACTING FERMIONS

In this section, we illustrate our discussion with several known examples.

A. $T=0$: Optimal (Lorentzian) pulses

An instructive example where the full-counting statistics is exactly computable is the case of several Lorentzian voltage pulses of one-quantum intensity each: $\int V(t) dt = \pm 2\pi\hbar$. In this case, as shown in Ref. 5, the full-counting statistics involves only a finite number of transferred electrons. For the effective-transparency spectrum, this implies a finite discrete spectrum with the number of levels equal to the number of pulses.^{5,21} In the special case when all pulses are of the same polarity (this also includes the setup of a constant voltage² as a limiting case), all the effective transparencies are degenerate and equal to the transparency of the junction g ,

$$\mu(p) = W\delta(p - g), \quad (17)$$

where $W = (2\pi\hbar)^{-1} \int V(t) dt$ is the number of pulses.

B. $T=0$: Periodic opening of a quantum point contact

An interesting exactly solvable example with a *continuous* spectrum of effective transparencies has been considered in Ref. 19: a periodic train of openings and closings of the contact. In this case, the distribution of effective transparencies is (Fig. 1)

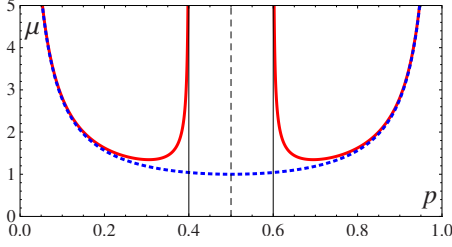


FIG. 1. (Color online) Effective-transparency distribution (18) for $g=1$ (blue dashed line) and $g=0.96$ (red solid line), plotted in the units of $4G/\pi^2$. For $g=0.96$, there is a spectral gap between $p=0.4$ and $p=0.6$.

$$\mu(p) = \frac{G}{\pi^2} \frac{\sqrt{g}}{p(1-p)} \operatorname{Re} \frac{|1-2p|}{\sqrt{(1-2p)^2 - (1-g)}}, \quad (18)$$

where the overall intensity [per opening/closing cycle, as in Eq. (4)] is

$$G = 2 \ln \frac{\sin \pi w}{\pi w_0}. \quad (19)$$

Here w is the fraction of a period during which the contact is open (with the transparency g) and w_0 is a regularization given, essentially, by the fraction of a period taken by the switching time.

One can see that if the contact is not fully open ($g < 1$), then there is a spectral gap around $\mu(p)=1/2$. We will comment on this feature in a future publication.²⁰

C. $T \neq 0$: Zero bias voltage

Remarkably, the same formula [Eq. (18)] gives the distribution of effective transparencies in the case of a finite temperature T (and zero bias voltage).^{4,19} The overall intensity is then given by

$$G = \frac{\pi T}{\hbar}, \quad (20)$$

and the observation time t is assumed to be much longer than \hbar/T , so that only the extensive part [Eq. (4)] is retained.

V. EXAMPLES: INTERACTING FERMIONS

The derivation of Sec. III shows that the absence of interactions between fermions leads to a generalized binomial statistics of the charge transfer between two conductors. In this section, we show that if the fermions interact, the zeros of the generating function $\chi(u)$ may shift to the complex plane.

Before presenting examples of full-counting statistics for interacting fermions, let us use a two-particle example to gain some intuition about the positions of the roots of $\chi(u)$. Consider a system of two particles which propagate from the left to the right lead of the contact. The full-counting statistics is then given by a quadratic polynomial $\chi(u) = P_0 + P_1 u + P_2 u^2$ with the probabilities $P_{0,1,2}$ obeying $P_0 + P_1 + P_2 = 1$. The roots of the polynomial become nonreal if

$$P_1^2 < 4P_0P_2 \quad \text{or, equivalently,} \quad \sqrt{P_0} + \sqrt{P_2} > 1, \quad (21)$$

i.e., if the probability P_1 of separating the two particles becomes too small. If one considers a strong attraction between particles, so that they become inseparable, then $P_1=0$, and the roots of $\chi(u)$ become purely imaginary. One thus sees that in this example (i) the nonreality of roots may be loosely related to an attraction between particles and (ii) a certain threshold of interaction may be needed to shift the roots of $\chi(u)$ from the real axis. We now turn to several more complicated examples of interacting systems where the full-counting statistics may be exactly computed.

A. Charge transfer in a normal metal–superconductor point contact

A more general example of an electronic system with attraction leading to charge transfer quantized in pairs of electrons [and hence to a shift of the zeros of $\chi(u)$ from the real axis] is considered in Ref. 22: a contact between a normal metal and a superconductor. In the low-temperature limit and at low bias voltage V (so that $T \ll eV \ll \Delta$), single-particle tunneling is suppressed, and the full-counting statistics is given by

$$\chi(u) = (1 - g_A + g_A u^2)^W. \quad (22)$$

Here g_A is the probability of Andreev reflection and $W = eVt/(2\pi\hbar)$ is the effective number of transfer attempts during the observation time t . Similarly to our two-particle example above, the zeros of $\chi(u)$ are in this case purely imaginary.

B. Two particles scattering on a resonant quantum dot

Another example of an interacting two-particle scattering problem has been studied in Ref. 23. In that work, one considers a quantum dot characterized by its resonances and by a Coulomb interaction between the particles. In this model, the scattering matrix is exactly computed and then used to find the full-counting statistics in a two-particle example. For the example considered in that work (a singlet two-particle wave packet in the shape of an exponentially truncated plane wave), the probabilities to transfer one and two electrons are given by (we refer the reader to Ref. 23 for details of the model and the derivation)

$$P_1 = \frac{2\beta}{(1+3\beta)(1+\beta)^2} \left[2 + 3\beta - \frac{1}{\delta^2 + 1} \right],$$

$$P_2 = \frac{1}{(1+3\beta)(1+\beta)^2} \left[1 + \frac{s\beta[4\delta^2 + 3(3+\beta)]}{2(1+\beta)(\delta^2 + 1)(\delta^2 + s^2)} \right], \quad (23)$$

where β is the dimensionless dwelling time on the dot characterizing the resonance, δ characterizes the strength of the Coulomb interaction [it is related to the parameter α introduced in Ref. 23 as $\delta = \alpha\beta/(1+\beta)$], and $s = (3+\beta)/[2(1+\beta)]$.

Using criterion (21), we can find the range of parameters δ and β where the statistics is generalized binomial (see Fig.

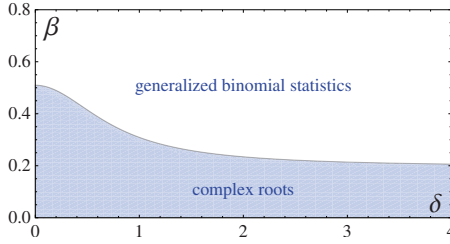


FIG. 2. (Color online) The shaded area shows the region in the parameter space of δ and β , where the roots of $\chi(u)$ are complex and the statistics is *not* generalized binomial.

2). Depending on the dwelling time, an interaction may either shift the roots of $\chi(\lambda)$ from the real axis or not.

Although this is a two-particle example, a qualitatively similar behavior of roots can also be found in multiparticle systems. For example, in Refs. 24 and 25, full-counting statistics has been calculated for charge and spin transport through a Kondo dot. In that problem, one can have both regimes with roots real and complex, depending on the parameters.

C. Charge pumping through a single-electron transistor

An interesting example with interaction where the statistics is nevertheless generalized binomial is studied in Ref. 26. The authors of that work consider a charge pump based on a nearly open quantum dot in the Coulomb blockade regime. In spite of the interaction on the dot, the problem is mapped onto an effective quadratic fermionic Hamiltonian (including a Majorana fermion representing the resonant level at the dot). Therefore, our formal conclusion about the generalized binomial statistics remains valid for this interacting system as well.

Indeed, one can easily verify that the generating function (per period) found in Ref. 26,

$$\ln \chi(\lambda) = -i\lambda + \frac{1}{2\hbar\omega} \int d\epsilon \times \ln \left(1 - \frac{\epsilon^2}{\epsilon^2 + \Gamma^2} \left[\frac{n_-(1-n_+)}{p} + \frac{n_+(1-n_-)}{1-p} \right] \right), \quad (24)$$

contains only real roots p (at any temperature). Here ω is the pumping frequency, $n_{\pm}(\epsilon) = (e^{(\epsilon \pm \hbar\omega)/T} + 1)^{-1}$ are the Fermi occupation numbers at the temperature T , Γ is an energy scale characterizing the strength of the Coulomb interaction in the dot, and $p = (1 - e^{i\lambda})^{-1}$, as usual.

In the zero-temperature limit, the effective-transparency distribution $\mu(p)$ can be easily calculated analytically,

$$\mu(p) = \begin{cases} \frac{\Gamma}{2\hbar\omega} p^{-1/2} (1-p)^{-3/2} & \text{for } p < \frac{(\hbar\omega)^2}{(\hbar\omega)^2 + \Gamma^2} \\ 0 & \text{for } p > \frac{(\hbar\omega)^2}{(\hbar\omega)^2 + \Gamma^2}. \end{cases} \quad (25)$$

The spectrum is continuous with a gap at $\frac{(\hbar\omega)^2}{(\hbar\omega)^2 + \Gamma^2} < p < 1$, as shown in Fig. 3 (one can check that the gap survives at finite

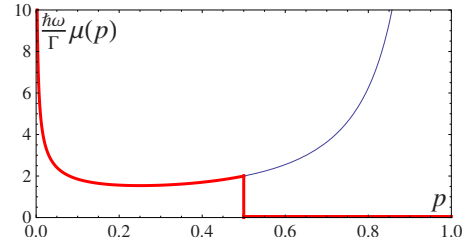


FIG. 3. (Color online) The effective-transparency distribution (25) in the units of $\Gamma/\hbar\omega$ (thick red line). $\mu(p)$ vanishes abruptly at $p = (\hbar\omega)^2 / [(\hbar\omega)^2 + \Gamma^2]$ (shown for $\hbar\omega = \Gamma$).

temperatures but changes its magnitude and shifts toward the center of the interval).

VI. CONCLUSION

To summarize, we have shown that the charge-transfer statistics for a system of noninteracting fermions is generalized binomial: it factorizes into individual single-particle tunneling events. This property holds under very general conditions: at arbitrary temperature and for an arbitrary time- and energy-dependent scatterer. Note that the converse is not true: an interacting system may either obey or disobey this factorization property, as we have seen in examples.

The factorization property may be formulated in terms of the positions of zeros of the generating function $\chi(u)$ [or, equivalently, of the singularities of $\ln \chi(u)$]. The factorization implies that those singularities are confined to the negative real axis. In this case one can fully characterize full-counting statistics by the distribution of effective transparencies $\mu(p)$. All charge-transfer properties (moments and cumulants of the transferred charge, entanglement between the leads,¹⁹ etc.) can be expressed in terms of $\mu(p)$.

The absence of factorization corresponds to a shift of singularities from the real axis. In this case, $\chi(u)$ can still be characterized by its zeros albeit in the complex u plane. This description is somewhat similar to the approach of Lee and Yang²⁷ in statistical physics, where a partition function is characterized by the zeros of its analytic extension to complex values of physical parameters such as magnetic field or chemical potential. In statistical physics, this approach turned out to be very useful for understanding possible phase transitions: a phase transition is determined by the location of zeros of the partition function relative to the real (physical) axis of parameters.²⁷ From this perspective, there remain interesting questions of characterizing interactions in the scattering problem by the positions of singularities of $\ln \chi(u)$ and of understanding possible physical implications of their shift away from the real axis.

As an interesting extension of our results, one can consider constraints on full-counting statistics for noninteracting electrons in a multiterminal setup. In this case, a certain “reality condition” on the zeros of the generating function is possible.²⁸ However, the factorization property does not hold, being specific to the two-terminal case.

Another possible generalization of our construction is to use the formalism developed in Secs. II and III for other

types of problems with quadratic Hamiltonians, not necessarily involving physical charge transfer. As an example of such an application, one can study the statistics of staggered magnetization of an XY spin chain (mappable to free fermions by a Jordan-Wigner transformation). See also Ref. 29 for applications of those ideas to analyzing the entropy in such systems.

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APPENDIX A: FERMIONIC BILINEARS AND TRACE IDENTITIES

For reader's convenience, we present in this appendix some helpful identities on the exponentials of fermionic bilinears. Those identities are then used to derive the determinant formula [Eq. (5)]. A more detailed account including proofs may be found in Ref. 11.

For any single-particle observable A , the corresponding multiparticle operator acting in the Fock space is given by the *fermionic bilinear*,

$$\hat{A} = \psi^\dagger A \psi. \quad (\text{A1})$$

This construction provides a representation of the $gl(n)$ algebra in the 2^n -dimensional Fock space (n denotes here the dimension of the single-particle Hilbert space). Indeed, one easily checks that commutators of fermionic bilinears are given by

$$[\hat{A}, \hat{B}] = \widehat{[A, B]}. \quad (\text{A2})$$

Exponentiating these relations, one obtains a representation of the group $GL(n)$,³⁰

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{C}} \quad \text{if} \quad e^A e^B = e^C. \quad (\text{A3})$$

One can use this representation to show that the evolution operator for a system of noninteracting fermions is an exponential of a fermionic bilinear. Indeed, the multiparticle evolution operator is given by the time-ordered exponential

$$\hat{U} = \text{T exp} \left[-i \int \hat{H}(t) dt \right], \quad (\text{A4})$$

where the time-dependent Hamiltonian \hat{H} is a fermionic bilinear. This time-ordered exponential can be represented as a chronologically ordered product of (infinitesimal) exponents of fermionic bilinears and, by virtue of Eq. (A3), may be rewritten as a single exponential

$$\hat{U} = \exp[\psi^\dagger (\ln U) \psi], \quad (\text{A5})$$

where

$$U = \text{T exp} \left[-i \int H(t) dt \right] \quad (\text{A6})$$

is the *single-particle* evolution operator.

For traces of such exponentials one has a simple formula,¹¹

$$\text{Tr} e^{\hat{A}} = \det(1 + e^A), \quad (\text{A7})$$

where the trace is taken in the Fock space, and the determinant is in the single-particle space. For traces of products of exponentials of fermionic bilinears, one finds, using Eqs. (A3) and (A7),

$$\text{Tr}(e^{\hat{A}_1} e^{\hat{A}_2} \dots e^{\hat{A}_k}) = \det(1 + e^{A_1} e^{A_2} \dots e^{A_k}). \quad (\text{A8})$$

From this relation, a derivation of the Levitov-Lesovik determinant formula easily follows

$$\begin{aligned} \frac{\text{Tr}(\hat{\rho}_0 \hat{U}^\dagger e^{i\lambda \hat{Q}} \hat{U} e^{-i\lambda \hat{Q}})}{\text{Tr} \hat{\rho}_0} &= \frac{\det(1 + e^{-h_0} U^\dagger e^{i\lambda Q} U e^{-i\lambda Q})}{\det(1 + e^{-h_0})} \\ &= \det[1 + n_F(U^\dagger e^{i\lambda Q} U e^{-i\lambda Q} - 1)], \end{aligned} \quad (\text{A9})$$

where n_F is defined by Eq. (6).

APPENDIX B: ADIABATIC AND INSTANT-SCATTERING APPROXIMATIONS

Determinant formula (5) has been derived under the very general assumptions specified in Sec. II. A physically relevant and technically simpler approximation is possible in the situation when the two leads involve asymptotically free electrons, so that the scattering may be described in terms of an instantaneous scattering matrix $S(t, \epsilon)$. This matrix relates the asymptotic states for electrons at energy ϵ scattering on the instantaneous Hamiltonian $H(t)$. If the Hamiltonian varies slowly on the scale of the ‘‘scattering time’’ $\hbar \partial \log S / \partial \epsilon$, then the evolution operator U may be approximated by the Wigner transformation of $S(t, \epsilon)$.¹⁴ The corresponding *adiabaticity condition* may be formulated as

$$\hbar \left| \frac{\partial S^{-1}}{\partial t} \frac{\partial S}{\partial \epsilon} \right| \ll 1. \quad (\text{B1})$$

At zero temperature, this condition allows to neglect the energy dependence of $S(t, \epsilon)$ and to replace the scattering operator U in Eq. (5) by the instantaneous scattering matrix $S(t)$ taken at the Fermi energy ϵ (since only the states in the vicinity of the Fermi energy contribute to the determinant).^{3,5,14}

At finite temperature, to neglect the energy dependence of the scattering matrix, one also needs that the energy dependence of S is small on the scale determined by the temperature. In addition to adiabaticity condition (B1) we require

$$T \left| S^{-1} \frac{\partial S}{\partial \epsilon} \right| \ll 1. \quad (\text{B2})$$

The two conditions [Eqs. (B1) and (B2)] constitute the *instant-scattering* approximation (equivalent to the adiabatic approximation in the case of zero temperature). In this approximation, the determinant formula becomes

$$\chi(\lambda) = \det(1 + n_F(S^\dagger e^{i\lambda Q} S e^{-i\lambda Q} - 1)), \quad (\text{B3})$$

where $S(t)$ is a local in time unitary scattering matrix.

Determinant (B3), in turn, requires a regularization at negative energies since the operator under the determinant tends at infinite negative energies to $S^\dagger e^{i\lambda Q} S e^{-i\lambda Q}$, a nonunity matrix. This regularization may be performed either by taking into account the finiteness of the negative-energy spectrum or by reintroducing a weak energy dependence of $S(t, \epsilon)$ so that $\lim_{\epsilon \rightarrow -\infty} S(t, \epsilon) = 1$. In the latter case, the easiest way to implement the regularization is to rewrite Eq. (5) as¹⁵

$$\chi(\lambda) = \det([1 + n_F(U^\dagger e^{i\lambda Q} U e^{-i\lambda Q} - 1)]e^{i\lambda n_F Q} U^\dagger e^{-i\lambda n_F Q} U). \quad (\text{B4})$$

This manipulation is obviously admissible for the “full” evolution operator U corresponding to the regularized scattering matrix with $\lim_{\epsilon \rightarrow -\infty} S(t, \epsilon) = 1$. However, in new expression (B4), one can simply replace U by a local in time scattering matrix $S(t)$ to obtain a fully convergent expression in the instant-scattering limit.

Regularized formula (B4) proposed in Ref. 15 is equivalent to the previously existing regularization prescriptions. Our discussion in this paper does not rely on this regularization since we always consider the general case of an arbitrary noninteracting evolution operator and therefore may assume that the negative-energy asymptotics is suitably regularized.

APPENDIX C: SPECTRUM OF THE EFFECTIVE TRANSPARENCIES IN THE ADIABATIC LIMIT AT ZERO TEMPERATURE

Generalized binomial statistics for noninteracting fermions has already been proven in our earlier publication under

the assumption of zero temperature and instant scattering.¹⁰ Of course, that result now follows from the more general argument of the present work. However, we find it instructive to demonstrate explicitly the equivalence of the two results under those more restrictive assumptions.

It has been shown in Ref. 10 that, at zero temperature and in the instant-scattering limit, the effective transparencies of elementary single-particle events are given by the spectrum of the operator

$$n_F^z = Q S n_F S^\dagger Q \quad (\text{C1})$$

(the same operator was denoted M in Ref. 19). In this appendix, we explicitly show that the spectra of n_F^z and of X coincide, except for the eigenvalues zero and one.

Indeed, at zero temperature n_F is a projector operator, and therefore the “hermitized” operator \tilde{X} given by Eq. (9), whose spectrum coincides with the spectrum of X , is block diagonal. The block at positive energies equals Q and only gives eigenvalues zero and one. The block at negative energies can be written (at zero temperature) as

$$\tilde{X}_- = n_F S^\dagger Q S n_F = R^\dagger R, \quad \text{where } R = Q S n_F \quad (\text{C2})$$

(we have also used that Q is a projector). On the other hand, one can rewrite (C1) as

$$n_F^z = R R^\dagger \quad (\text{C3})$$

from where it becomes obvious that the spectra of \tilde{X}_- and n_F^z coincide, except for zero eigenvalues (their eigenvectors can be related to each other by the operators R and R^\dagger). This completes the proof of the coincidence of the spectra of X and n_F^z (modulo eigenvalues zero and one, which do not contribute to the noise, but only to the overall deterministic charge transfer).

¹Ya. M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).

²L. S. Levitov and G. B. Lesovik, Pis'ma Zh. Eksp. Teor. Fiz. **58**, 225 (1993) [JETP Lett. **58**, 230 (1993)].

³D. A. Ivanov and L. S. Levitov, Pis'ma Zh. Eksp. Teor. Fiz. **58**, 450 (1993) [JETP Lett. **58**, 461 (1993). The reader should be careful about an excessive number of typos in the English translation].

⁴L. S. Levitov, H.-W. Lee, and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).

⁵D. A. Ivanov, H.-W. Lee, and L. S. Levitov, Phys. Rev. B **56**, 6839 (1997).

⁶L. S. Levitov, arXiv:cond-mat/0103617 (unpublished).

⁷M. Vanević, Yu. V. Nazarov, and W. Belzig, Phys. Rev. B **78**, 245308 (2008).

⁸F. Hassler, M. V. Suslov, G. M. Graf, M. V. Lebedev, G. B. Lesovik, and G. Blatter, Phys. Rev. B **78**, 165330 (2008).

⁹M. Vanević, Yu. V. Nazarov, and W. Belzig, Phys. Rev. Lett. **99**, 076601 (2007).

¹⁰A. G. Abanov and D. A. Ivanov, Phys. Rev. Lett. **100**, 086602 (2008).

¹¹I. Klich, in *Quantum Noise in Mesoscopic Physics*, edited by Yu.

V. Nazarov (Springer, New York, 2003).

¹²M. Büttiker, Phys. Rev. B **46**, 12485 (1992).

¹³P. W. Brouwer, Phys. Rev. B **58**, R10135 (1998).

¹⁴B. A. Muzykantskii and Y. Adamov, Phys. Rev. B **68**, 155304 (2003).

¹⁵J. E. Avron, S. Bachmann, G. M. Graf, and I. Klich, Commun. Math. Phys. **280**, 807 (2008).

¹⁶The operator $(n_F)^{-1/2}$ is well defined only at finite temperatures.

However, one easily sees that the spectra of X and \tilde{X} coincide even at zero temperature by taking the corresponding limit.

¹⁷We use the fact that $\text{tr} \log(AB) = \text{tr} \log A + \text{tr} \log B$ if both terms are finite, i.e., if the operators $\log A$ and $\log B$ are of trace class.

¹⁸Note that our results do not contradict those in Refs. 7 and 9 where charge transport at zero temperature in the bias-voltage setup has been factorized into one- and two-particle processes. Indeed, the two-particle (“bidirectional”) processes in those works can be further factorized into two symmetric single-particle transfers, in agreement with our result.

¹⁹I. Klich and L. Levitov, Phys. Rev. Lett. **102**, 100502 (2009).

²⁰A. G. Abanov and D. A. Ivanov (unpublished).

²¹Y. B. Sherkunov, A. Pratap, B. Muzykantskii, and N.

- d'Ambrumenil, Phys. Rev. Lett. **100**, 196601 (2008).
- ²²B. A. Muzykantskii and D. E. Khmel'nitskii, Phys. Rev. B **50**, 3982 (1994).
- ²³A. V. Lebedev, G. B. Lesovik, and G. Blatter, Phys. Rev. Lett. **100**, 226805 (2008).
- ²⁴A. Komnik and A. O. Gogolin, Phys. Rev. Lett. **94**, 216601 (2005).
- ²⁵T. L. Schmidt, A. O. Gogolin, and A. Komnik, Phys. Rev. B **75**, 235105 (2007).
- ²⁶A. V. Andreev and E. G. Mishchenko, Phys. Rev. B **64**, 233316 (2001).
- ²⁷C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952); T. D. Lee and C. N. Yang, *ibid.* **87**, 410 (1952).
- ²⁸D. Kambly, Master thesis, EPFL, Lausanne, 2009.
- ²⁹I. Klich, J. Phys. A **39**, L85 (2006); I. Klich, G. Refael, and A. Silva, Phys. Rev. A **74**, 032306 (2006).
- ³⁰One can prove Eq. (A3) by using Baker-Campbell-Hausdorff formula. See, e.g., Yu. A. Bakhturin, in *Encyclopedia of Mathematics*, edited by M. Hazewinkel (Kluwer, Dordrecht, 2002).